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Citation for final published version:

Aliev, Iskander ORCID: <https://orcid.org/0000-0002-2206-9207>, Averkov, Gennadiy, De Lora, Jesus A. and Oertel, Timm ORCID: <https://orcid.org/0000-0001-5720-8978> 2020. Optimizing sparsity over lattices and semigroups. Lecture Notes in Computer Science file

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Optimizing Sparsity over Lattices and Semigroups

Iskander Aliev¹, Gennadiy Averkov², Jesús A. De Loera³, and Timm Oertel¹

¹ Cardiff University, UK

² Brandenburg University of Technology Cottbus-Senftenberg, Germany

³ University of California, Davis, USA

Abstract. Motivated by problems in optimization we study the *sparsity* of the solutions to systems of linear Diophantine equations and linear integer programs, i.e., the number of non-zero entries of a solution, which is often referred to as the ℓ_0 -norm. Our main results are improved bounds on the ℓ_0 -norm of sparse solutions to systems $A\mathbf{x} = \mathbf{b}$, where $A \in \mathbb{Z}^{m \times n}$, $\mathbf{b} \in \mathbb{Z}^m$ and \mathbf{x} is either a general integer vector (lattice case) or a non-negative integer vector (semigroup case). In the lattice case and certain scenarios of the semigroup case, we give polynomial time algorithms for computing solutions with ℓ_0 -norm satisfying the obtained bounds.

1 Introduction

This paper discusses the problem of finding sparse solutions to systems of linear Diophantine equations and integer linear programs. We investigate the ℓ_0 -norm $\|\mathbf{x}\|_0 := |\{i : x_i \neq 0\}|$, a function widely used in the theory of *compressed sensing* [6,9], which measures the sparsity of a given vector $\mathbf{x} = (x_1, \dots, x_n)^\top \in \mathbb{R}^n$ (it is clear that the ℓ_0 -norm is actually not a norm).

Sparsity is a topic of interest in several areas of optimization. The ℓ_0 -norm minimization problem over reals is central in the theory of the classical compressed sensing, where a linear programming relaxation provides a guaranteed approximation [8,9]. Support minimization for solutions to Diophantine equations is relevant for the theory of compressed sensing for discrete-valued signals [11,12,17]. There is still little understanding of discrete signals in the compressed sensing paradigm, despite the fact that there are many applications in which the signal is known to have discrete-valued entries, for instance, in wireless communication [22] and the theory of error-correcting codes [7]. Sparsity was also investigated in integer optimization [1,10,20], where many combinatorial optimization problems have useful interpretations as sparse semigroup problems. For example, the edge-coloring problem can be seen as a problem in the semigroup generated by matchings of the graph [18]. Our results provide natural out-of-the-box sparsity bounds for problems with linear constraints and integer variables in a general form.

1.1 Lattices: sparse solutions of linear Diophantine systems

Each integer matrix $A \in \mathbb{Z}^{m \times n}$ determines the lattice $\mathcal{L}(A) := \{A\mathbf{x} : \mathbf{x} \in \mathbb{Z}^n\}$ generated by the columns of A . By an easy reduction via row transformations, we may assume without loss of generality that the rank of A is m .

Let $[n] := \{1, \dots, n\}$ and let $\binom{[n]}{m}$ be the set of all m -element subsets of $[n]$. For $\gamma \subseteq [n]$, consider the $m \times |\gamma|$ submatrix A_γ of A with columns indexed by γ . One can easily prove that the determinant of $\mathcal{L}(A)$ is equal to

$$\gcd(A) := \gcd \left\{ \det(A_\gamma) : \gamma \in \binom{[n]}{m} \right\}.$$

Since $\mathcal{L}(A_\gamma)$ is the lattice spanned by the columns of A indexed by γ , it is a sublattice of $\mathcal{L}(A)$. We first deal with a natural question: *Can the description of a given lattice $\mathcal{L}(A)$ in terms of A be made sparser by passing from A to A_γ with γ having a smaller cardinality than n and satisfying $\mathcal{L}(A) = \mathcal{L}(A_\gamma)$?* That is, we want to discard some of the columns of A and generate $\mathcal{L}(A)$ by $|\gamma|$ columns with $|\gamma|$ being possibly small.

For stating our results, we need several number-theoretic functions. Given $z \in \mathbb{Z}_{>0}$, consider the prime factorization $z = p_1^{s_1} \cdots p_k^{s_k}$ with pairwise distinct prime factors p_1, \dots, p_k and their multiplicities $s_1, \dots, s_k \in \mathbb{Z}_{>0}$. Then the number of prime factors $\sum_{i=1}^k s_i$ counting the multiplicities is denoted by $\Omega(z)$. Furthermore, we introduce $\Omega_m(z) := \sum_{i=1}^k \min\{s_i, m\}$. That is, by introducing m we set a threshold to account for multiplicities. In the case $m = 1$ we thus have $\omega(z) := \Omega_1(z) = k$, which is the number of prime factors in z , not taking the multiplicities into account. The functions Ω and ω are called *prime Ω -function* and *prime ω -function*, respectively, in number theory [15]. We call Ω_m the *truncated prime Ω -function*.

Theorem 1 *Let $A \in \mathbb{Z}^{m \times n}$, with $m \leq n$, and let $\tau \in \binom{[n]}{m}$ be such that the matrix A_τ is non-singular. Then the equality $\mathcal{L}(A) = \mathcal{L}(A_\gamma)$ holds for some γ satisfying $\tau \subseteq \gamma \subseteq [n]$ and*

$$|\gamma| \leq m + \Omega_m \left(\frac{|\det(A_\tau)|}{\gcd(A)} \right). \quad (1)$$

Given A and τ , the set γ can be computed in polynomial time.

One can easily see that $\omega(z) \leq \Omega_m(z) \leq \Omega(z) \leq \log_2(z)$ for every $z \in \mathbb{Z}_{>0}$. The estimate using $\log_2(z)$ gives a first impression on the quality of the bound (1). It turns out, however, that $\Omega_m(z)$ is much smaller on the average. Results in number theory [15, §22.10] show that the average values $\frac{1}{z}(\omega(1) + \cdots + \omega(z))$ and $\frac{1}{z}(\Omega(1) + \cdots + \Omega(z))$ are of order $\log \log z$, as $z \rightarrow \infty$.

As an immediate consequence of Theorem 1 we obtain

Corollary 2 *Consider the linear Diophantine system*

$$A\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \in \mathbb{Z}^n \quad (2)$$

with $A \in \mathbb{Z}^{m \times n}$, $\mathbf{b} \in \mathbb{Z}^m$ and $m \leq n$. Let $\tau \in \binom{[n]}{m}$ be such that the $m \times m$ matrix A_τ is non-singular. If (2) is feasible, then (2) has a solution \mathbf{x} satisfying the sparsity bound

$$\|\mathbf{x}\|_0 \leq m + \Omega_m \left(\frac{|\det(A_\tau)|}{\gcd(A)} \right).$$

Under the above assumptions, for given A, \mathbf{b} and τ , such a sparse solution can be computed in polynomial time.

From the optimization perspective, Corollary 2 deals with the problem

$$\min \{ \|\mathbf{x}\|_0 : A\mathbf{x} = \mathbf{b}, \mathbf{x} \in \mathbb{Z}^n \}$$

of minimization of the ℓ_0 -norm over the affine lattice $\{\mathbf{x} \in \mathbb{Z}^n : A\mathbf{x} = \mathbf{b}\}$.

1.2 Semigroups: sparse solutions in integer programming

Consider next the standard form of the feasibility constraints of integer linear programming

$$A\mathbf{x} = \mathbf{b}, \mathbf{x} \in \mathbb{Z}_{\geq 0}^n. \quad (3)$$

For a given matrix A , the set of all \mathbf{b} such that (3) is feasible, is the *semigroup* $\mathcal{Sg}(A) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{Z}_{\geq 0}^n\}$ generated by the columns of A .

If (3) has a solution, i.e., $\mathbf{b} \in \mathcal{Sg}(A)$, how sparse can such a solution be? In other words, we are interested in the ℓ_0 -norm minimization problem

$$\min \{ \|\mathbf{x}\|_0 : A\mathbf{x} = \mathbf{b}, \mathbf{x} \in \mathbb{Z}_{\geq 0}^n \}. \quad (4)$$

It is clear that Problem (4) is NP-hard, because deciding the feasibility of (3) [23, § 18.2] or even solving the relaxation of (4) with the condition $\mathbf{x} \in \mathbb{Z}_{\geq 0}^n$ replaced by $\mathbf{x} \in \mathbb{R}^n$ [19] is NP-hard.

Taking the NP-hardness of Problem (4) into account, our aim is to *estimate* the optimal value of (4) under the assumption that this problem is feasible. In [2, Theorem 1.1 (i)] (see also [1, Theorem 1]), it was shown that for any $\mathbf{b} \in \mathcal{Sg}(A)$, there exists a $\mathbf{x} \in \mathbb{Z}^n$, such that $A\mathbf{x} = \mathbf{b}$ and

$$\|\mathbf{x}\|_0 \leq m + \left\lceil \log_2 \left(\frac{\sqrt{\det(AA^\top)}}{\gcd(A)} \right) \right\rceil. \quad (5)$$

In [1, Theorem 2], it was shown that Equation (5) cannot be improved significantly, but nevertheless we show here how to improve it in some special cases. As a consequence of Theorem 1 we obtain the following.

Corollary 3 *Let $A \in \mathbb{Z}^{m \times n}$ be a matrix whose columns positively span \mathbb{R}^m and let $\mathbf{b} \in \mathbb{Z}^m$. Then $\mathcal{L}(A) = \mathcal{Sg}(A)$. Furthermore, if $\mathbf{b} \in \mathcal{L}(A)$, and $\tau \in \binom{[n]}{m}$ is a set, for which the matrix A_τ is non-singular, then there is a solution \mathbf{x} of*

the integer-programming feasibility problem $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \in \mathbb{Z}_{\geq 0}^m$ that satisfies the sparsity bound

$$\|\mathbf{x}\|_0 \leq 2m + \Omega_m \left(\frac{|\det(A_\tau)|}{\gcd(A)} \right). \quad (6)$$

Under the above assumptions, for given A , \mathbf{b} and τ , such a sparse solution \mathbf{x} can be computed in polynomial time.

Note that for a fixed m , (6) is usually much tighter than (5), because the function $\Omega_m(z)$ is bounded from above by the logarithmic function $\log_2(z)$ and is much smaller than $\log_2(z)$ on the average. Furthermore, $|\det(A_\tau)| \leq \sqrt{\det(AA^\top)}$ in view of the Cauchy-Binet formula.

We take a closer look at the case $m = 1$ of a single equation and tighten the given bounds in this case. That is, we consider the *knapsack feasibility problem*

$$\mathbf{a}^\top \mathbf{x} = b, \quad \mathbf{x} \in \mathbb{Z}_{\geq 0}^n, \quad (7)$$

where $\mathbf{a} \in \mathbb{Z}^n$ and $b \in \mathbb{Z}$. Without loss of generality we can assume that all components of the vector \mathbf{a} are not equal to zero. It follows from (5) that a feasible problem (7) has a solution \mathbf{x} with

$$\|\mathbf{x}\|_0 \leq 1 + \left\lfloor \log \left(\frac{\|\mathbf{a}\|_2}{\gcd(\mathbf{a})} \right) \right\rfloor. \quad (8)$$

If all components of \mathbf{a} have the same sign, without loss of generality we can assume $\mathbf{a} \in \mathbb{Z}_{>0}^n$. In this setting, Theorem 1.2 in [2] strengthens the bound (8) by replacing the ℓ_2 -norm of the vector \mathbf{a} with the ℓ_∞ -norm. It was conjectured in [2, page 247] that a bound $\|\mathbf{x}\|_0 \leq c + \lfloor \log_2 (\|\mathbf{a}\|_\infty / \gcd(\mathbf{a})) \rfloor$ with an absolute constant c holds for an arbitrary $\mathbf{a} \in \mathbb{Z}^n$. We obtain the following result, which covers the case that has not been settled so far and yields a confirmation of this conjecture.

Corollary 4 *Let $\mathbf{a} = (a_1, \dots, a_n)^\top \in (\mathbb{Z} \setminus \{0\})^n$ be a vector that contains both positive and negative components. If the knapsack feasibility problem $\mathbf{a}^\top \mathbf{x} = b$, $\mathbf{x} \in \mathbb{Z}_{\geq 0}^n$ has a solution, then there is a solution \mathbf{x} satisfying the sparsity bound*

$$\|\mathbf{x}\|_0 \leq 2 + \min \left\{ \omega \left(\frac{|a_i|}{\gcd(\mathbf{a})} \right) : i \in [n] \right\}.$$

Under the above assumptions, for given \mathbf{a} and b , such a sparse solution \mathbf{x} can be computed in polynomial time.

Our next contribution is that, given additional structure on A , we can improve on [2, Theorem 1.1 (i)], which in turn also gives an improvement on [2, Theorem 1.2]. For $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{R}^m$, we denote by $\text{cone}(\mathbf{a}_1, \dots, \mathbf{a}_n)$ the convex conic hull of the set $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$. Now assume the matrix $A = (\mathbf{a}_1, \dots, \mathbf{a}_n) \in \mathbb{Z}^{m \times n}$ with columns \mathbf{a}_i satisfies the following conditions:

$$\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{Z}^m \setminus \{\mathbf{0}\}, \quad (9)$$

$$\text{cone}(\mathbf{a}_1, \dots, \mathbf{a}_n) \text{ is an } m\text{-dimensional pointed cone}, \quad (10)$$

$$\text{cone}(\mathbf{a}_1) \text{ is an extreme ray of } \text{cone}(\mathbf{a}_1, \dots, \mathbf{a}_n). \quad (11)$$

Note that the previously best sparsity bound for the general case of the integer-programming feasibility problem is (5). Using the Cauchy-Binet formula, (5) can be written as

$$\|\mathbf{x}\|_0 \leq m + \log_2 \frac{\sqrt{\sum_{I \in \binom{[n]}{m}} \det(A_I)^2}}{\gcd(A)}.$$

The following theorem improves this bound in the “*pointed cone case*” by removing a fraction of m/n of terms in the sum under the square root.

Theorem 5 *Let $A = (\mathbf{a}_1, \dots, \mathbf{a}_n) \in \mathbb{Z}^{m \times n}$ satisfy (9)–(11) and, for $\mathbf{b} \in \mathbb{Z}^m$, consider the integer-programming feasibility problem*

$$A\mathbf{x} = \mathbf{b}, \mathbf{x} \in \mathbb{Z}_{\geq 0}^n. \quad (12)$$

If (12) is feasible, then there is a feasible solution \mathbf{x} satisfying the sparsity bound

$$\|\mathbf{x}\|_0 \leq m + \left\lfloor \log_2 \frac{q(A)}{\gcd(A)} \right\rfloor,$$

where

$$q(A) := \sqrt{\sum_{I \in \binom{[n]}{m} : 1 \in I} \det(A_I)^2}.$$

We omit the proof of this result due to the page limit for the IPCO proceedings. Instead we focus on the particularly interesting case $m = 1$. In this case, assumption (10) is equivalent to $\mathbf{a} \in \mathbb{Z}_{>0}^n \cup \mathbb{Z}_{<0}^n$. Without loss of generality, one can assume $\mathbf{a} \in \mathbb{Z}_{>0}^n$.

Theorem 6 *Let $\mathbf{a} = (a_1, \dots, a_n)^\top \in \mathbb{Z}_{>0}^n$ and $b \in \mathbb{Z}_{\geq 0}$. If the knapsack feasibility problem $\mathbf{a}^\top \mathbf{x} = b$, $\mathbf{x} \in \mathbb{Z}_{\geq 0}^n$ has a solution, there is a solution \mathbf{x} satisfying the sparsity bound*

$$\|\mathbf{x}\|_0 \leq 1 + \left\lfloor \log_2 \left(\frac{\min\{a_1, \dots, a_n\}}{\gcd(\mathbf{a})} \right) \right\rfloor.$$

When dealing with bounds for sparsity it would be interesting to understand *the worst case scenario among all members of the semigroup*, which is described by the function

$$\text{ICR}(A) = \max_{\mathbf{b} \in \text{Sg}(A)} \min\{\|\mathbf{x}\|_0 : A\mathbf{x} = \mathbf{b}, \mathbf{x} \in \mathbb{Z}_{\geq 0}^n\}. \quad (13)$$

We call $\text{ICR}(A)$ the *integer Carathéodory rank* in resemblance to the classical problem of finding the integer Carathéodory number for Hilbert bases [24]. Above results for the problem $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \in \mathbb{Z}_{\geq 0}^n$ can be phrased as upper bounds on $\text{ICR}(A)$. We are interested in the complexity of computing $\text{ICR}(A)$. The first question is: *can the integer Carathéodory rank of a matrix A be computed at all?* After all, remember that the semigroup has infinitely many elements

and, despite the fact that $\text{ICR}(A)$ is a finite number, a direct usage of (13) would result into the determination of the sparsest representation $A\mathbf{x} = \mathbf{b}$ for all of the infinitely many elements \mathbf{b} of $\mathcal{S}g(A)$. It turns out that $\text{ICR}(A)$ is computable, as the inequality $\text{ICR}(A) \leq k$ can be expressed as the formula $\forall \mathbf{x} \in \mathbb{Z}_{\geq 0}^n \exists \mathbf{y} \in \mathbb{Z}_{\geq 0}^n : (A\mathbf{x} = A\mathbf{y}) \wedge (\|\mathbf{y}\|_0 \leq k)$ in *Presburger arithmetic* [14]. Beyond this fact, the complexity status of computing $\text{ICR}(A)$ is largely open, even when A is just one row:

Problem 7 *Given the input $\mathbf{a} = (a_1, \dots, a_n)^\top \in \mathbb{Z}^n$, is it NP-hard to compute $\text{ICR}(\mathbf{a}^\top)$?*

The *Frobenius number* $\max \mathbb{Z}_{\geq 0} \setminus \mathcal{S}g(\mathbf{a}^\top)$, defined under the assumptions $\mathbf{a} \in \mathbb{Z}_{>0}^n$ and $\gcd(\mathbf{a}) = 1$, is yet another value associated to $\mathcal{S}g(\mathbf{a}^\top)$. The Frobenius number can be computed in polynomial time when n is fixed [5,16] but is NP-hard to compute when n is not fixed [21]. It seems that there might be a connection between computing the Frobenius number and $\text{ICR}(\mathbf{a}^\top)$.

2 Proofs of Theorem 1 and its consequences

The proof of Theorem 1 relies on the theory of finite Abelian groups. We write Abelian groups additively. An Abelian group G is said to be a *direct sum* of its finitely many subgroups G_1, \dots, G_m , which is written as $G = \bigoplus_{i=1}^m G_i$, if every element $x \in G$ has a unique representation as $x = x_1 + \dots + x_m$ with $x_i \in G_i$ for each $i \in [m]$. A *primary cyclic group* is a non-zero finite cyclic group whose order is a power of a prime number. We use G/H to denote the quotient of G modulo its subgroup H .

The fundamental theorem of finite Abelian groups states that every finite Abelian group G has a *primary decomposition*, which is essentially unique. This means, G is decomposable into a direct sum of its primary cyclic groups and that this decomposition is unique up to automorphisms of G . We denote by $\kappa(G)$ the number of direct summands in the primary decomposition of G .

For a subset S of a finite Abelian group G , we denote by $\langle S \rangle$ the subgroup of G generated by S . We call a subset S of G *non-redundant* if the subgroups $\langle T \rangle$ generated by proper subsets T of S are properly contained in $\langle S \rangle$. In other words, S is non-redundant if $\langle S \setminus \{x\} \rangle$ is a proper subgroup of $\langle S \rangle$ for every $x \in S$. The following result can be found in [13, Lemma A.6].

Theorem 8 *Let G be a finite Abelian group. Then the maximum cardinality of a non-redundant subset S of G is equal to $\kappa(G)$.*

We will also need the following lemmas, proved in the Appendix.

Lemma 1. *Let G be a finite Abelian group representable as a direct sum $G = \bigoplus_{j=1}^m G_j$ of $m \in \mathbb{Z}_{>0}$ cyclic groups. Then $\kappa(G) \leq \Omega_m(|G|)$.*

Lemma 2. *Let Λ be a sublattice of \mathbb{Z}^m of rank $m \in \mathbb{Z}_{>0}^m$. Then $G = \mathbb{Z}^m / \Lambda$ is a finite Abelian group of order $\det(\Lambda)$ that can be represented as a direct sum of at most m cyclic groups.*

Proof (Theorem 1). Let $\mathbf{a}_1, \dots, \mathbf{a}_n$ be the columns of A . Without loss of generality, let $\tau = [m]$. We use the notation $B := A_\tau$.

Reduction to the case $\gcd(A) = 1$. For a non-singular square matrix M , the columns of $M^{-1}A$ are representations of the columns of A in the basis of columns of M . In particular, for a matrix M whose columns form a basis of $\mathcal{L}(A)$, the matrix $M^{-1}A$ is integral and the $m \times m$ minors of $M^{-1}A$ are the respective $m \times m$ minors of A divided by $\det(M) = \gcd(A)$. Thus, replacing A by $M^{-1}A$, we pass from $\mathcal{L}(A)$ to $\mathcal{L}(M^{-1}A) = \{M^{-1}z : z \in \mathcal{L}(A)\}$, which corresponds to a change of a coordinate system in \mathbb{R}^m and ensures that $\gcd(A) = 1$.

Sparsity bound (1). The matrix B gives rise to the lattice $\Lambda := \mathcal{L}(B)$ of rank m , while Λ determines the finite Abelian group \mathbb{Z}^m/Λ .

Consider the canonical homomorphism $\phi : \mathbb{Z}^m \rightarrow \mathbb{Z}^m/\Lambda$, sending an element of \mathbb{Z}^m to its coset modulo Λ . Since $\gcd(A) = 1$, we have $\mathcal{L}(A) = \mathbb{Z}^m$, which implies $\langle T \rangle = \mathbb{Z}^m/\Lambda$ for $T := \{\phi(\mathbf{a}_{m+1}), \dots, \phi(\mathbf{a}_n)\}$. For every non-redundant subset S of T , we have

$$\begin{aligned} |S| &\leq \kappa(\mathbb{Z}^m/\Lambda) && \text{(by Theorem 8)} \\ &\leq \Omega_m(|\det(A_\tau)|) && \text{(by Lemmas 1 and 2).} \end{aligned}$$

Fixing a set $I \subseteq \{m+1, \dots, n\}$ that satisfies $|I| = |S|$ and $S = \{\phi(\mathbf{a}_i) : i \in I\}$, we reformulate $\langle S \rangle = \mathbb{Z}^m/\Lambda$ as $\mathbb{Z}^m = \mathcal{L}(A_I) + \Lambda = \mathcal{L}(A_I) + \mathcal{L}(A_\tau) = \mathcal{L}(A_{I \cup \tau})$. Thus, (1) holds for $\gamma = I \cup \tau$.

Construction of γ in polynomial time. The matrix M used in the reduction to the case $\gcd(A) = 1$ can be constructed in polynomial time: one can obtain M from the Hermite Normal Form of A (with respect to the column transformations) by discarding zero columns. For the determination of γ , the set I that defines the non-redundant subset $S = \{\phi(\mathbf{a}_i) : i \in I\}$ of \mathbb{Z}^m/Λ needs to be determined. Start with $I = \{m+1, \dots, n\}$ and iteratively check if some of the elements $\phi(\mathbf{a}_i) \in \mathbb{Z}^m/\Lambda$, where $i \in I$, is in the group generated by the remaining elements. Suppose $j \in I$ and we want to check if $\phi(\mathbf{a}_j)$ is in the group generated by all $\phi(\mathbf{a}_i)$ with $i \in I \setminus \{j\}$. Since $\Lambda = \mathcal{L}(A_\tau)$, this is equivalent to checking $\mathbf{a}_j \in \mathcal{L}(A_{I \setminus \{j\} \cup \tau})$ and is thus reduced to solving a system of linear Diophantine equations with the left-hand side matrix $A_{I \setminus \{j\} \cup \tau}$ and the right-hand side vector \mathbf{a}_j . Thus, carrying the above procedure for every $j \in I$ and removing j from I whenever $\mathbf{a}_j \in \mathcal{L}(A_{I \setminus \{j\} \cup \tau})$, we eventually arrive at a set I that determines a non-redundant subset S of \mathbb{Z}^m/Λ . This is done by solving at most $n - m$ linear Diophantine systems in total, where the matrix of each system is a sub-matrix of A and the right-hand vector of the system is a column of A . \square

Remark 1 (Optimality of the bounds). For a given $\Delta \in \mathbb{Z}_{>2}$ let us consider matrices $A \in \mathbb{Z}^{m \times n}$ with $\Delta = |\det(A_\tau)|/\gcd(A)$. We construct a matrix A that shows the optimality of the bound (1). As in the proof of Theorem 1, we assume $\tau = [m]$ and use the notation $B = A_\tau$. Consider the prime factorization $\Delta = p_1^{n_1} \cdots p_s^{n_s}$. We will fix the matrix B to be a diagonal matrix with diagonal entries $d_1, \dots, d_m \in \mathbb{Z}_{>0}$ so that $\det(B) = d_1 \cdots d_m = \Delta$.

The diagonal entries are defined by distributing the prime factors of Δ among the diagonal entries of B . If the multiplicity n_i of the prime p_i is less than m ,

we introduce p_i as a factor of multiplicity 1 in n_i of the m diagonal entries of B . If the multiplicity n_i is at least m , we are able distribute the factors p_i among *all* of the diagonal entries of B so that each diagonal entry contains the factor p_i with multiplicity at least 1.

The group $\mathbb{Z}^m/\Lambda = \mathbb{Z}^m/\mathcal{L}(B)$ is a direct sum of m cyclic groups G_1, \dots, G_m of orders d_1, \dots, d_m , respectively. By the Chinese Remainder Theorem, these cyclic groups can be further decomposed into the direct sum of primary cyclic groups. By our construction, the prime factor p_i of the multiplicity $n_i < m$ generates a cyclic direct summand of order p_i in n_i of the subgroups G_1, \dots, G_m . If $n_i \geq m$, then each of the groups G_1, \dots, G_m has a direct summand, which is a non-trivial cyclic group whose order is a power of p_i . Summarizing, we see that the decomposition of \mathbb{Z}^m/Λ into primary cyclic groups contains n_i summands of order p_i , when $n_i < m$, and m summands, whose order is a power of p_i , when $n_i \geq m$. The total number of summands is thus $\sum_{i=1}^s \min\{m, n_i\} = \Omega_m(\Delta)$.

Now, fix $n = m + \Omega_m(\Delta)$ and choose columns $\mathbf{a}_{m+1}, \dots, \mathbf{a}_n$ so that $\phi(\mathbf{a}_{m+1}), \dots, \phi(\mathbf{a}_n)$ generate all direct summands in the decomposition of \mathbb{Z}^m/Λ into primary cyclic groups. With this choice, $\phi(\mathbf{a}_{m+1}), \dots, \phi(\mathbf{a}_n)$ generate \mathbb{Z}^m/Λ , which means that $\mathcal{L}(A) = \mathbb{Z}^m$ and implies $\gcd(A) = 1$. On the other hand, any proper subset $\{\phi(\mathbf{a}_{m+1}), \dots, \phi(\mathbf{a}_n)\}$ generates a proper subgroup of \mathbb{Z}^m/Λ , as some of the direct summands in the decomposition of \mathbb{Z}^m/Λ into primary cyclic groups will be missing. This means $\mathcal{L}(A_{[m] \cup I}) \subsetneq \mathbb{Z}^m$ for every $I \subsetneq \{m+1, \dots, n\}$.

Proof (Corollary 2). Feasibility of (2) can be expressed as $\mathbf{b} \in \mathcal{L}(A)$. Choose γ from the assertion of Theorem 1. One has $\mathbf{b} \in \mathcal{L}(A) = \mathcal{L}(A_\gamma)$ and so there exists a solution \mathbf{x} of (2) whose support is a subset of γ . This sparse solution \mathbf{x} can be computed by solving the Diophantine system with the left-hand side matrix A_γ and the right-hand side vector \mathbf{b} .

Proof (Corollary 3). Assume that the Diophantine system $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \in \mathbb{Z}^n$ has a solution. It suffices to show that, in this case, the integer-programming feasibility problem $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \in \mathbb{Z}_{\geq 0}^n$ has a solution, too, and that one can find a solution of the desired sparsity to the integer-programming feasibility problem in polynomial time.

One can determine γ as in Theorem 1 in polynomial time. Using γ , we can determine a solution $\mathbf{x}^* = (x_1^*, \dots, x_n^*)^\top \in \mathbb{Z}^n$ of the Diophantine system $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \in \mathbb{Z}^n$ satisfying $x_i^* = 0$ for $i \in [n] \setminus \gamma$ in polynomial time, as described in the proof of Corollary 2.

Let $\mathbf{a}_1, \dots, \mathbf{a}_n$ be the columns of A . Since the matrix A_τ is non-singular, the m vectors \mathbf{a}_i , where $i \in \tau$, together with the vector $\mathbf{v} = -\sum_{i \in \tau} \mathbf{a}_i$ positively span \mathbb{R}^n . Since all columns of A positive span \mathbb{R}^n , the conic version of the Carathéodory theorem implies the existence of a set $\beta \subseteq [m]$ with $|\beta| \leq m$, such that \mathbf{v} is in the conic hull of $\{\mathbf{a}_i : i \in \beta\}$. Consequently, the set $\{\mathbf{a}_i : i \in \beta \cup \tau\}$ and by this also the larger set $\{\mathbf{a}_i : i \in \beta \cup \gamma\}$ positively span \mathbb{R}^m . Let $I = \beta \cup \gamma$. By construction, $|I| \leq |\beta| + |\gamma| \leq m + |\gamma|$.

Since the vectors \mathbf{a}_i with $i \in I$ positively span \mathbb{R}^m , there exist a choice of rational coefficients $\lambda_i > 0$ ($i \in I$) with $\sum_{i \in I} \lambda_i \mathbf{a}_i = 0$. After rescaling we

can assume $\lambda_i \in \mathbb{Z}_{>0}$. Define $\mathbf{x}' = (x'_1, \dots, x'_n)^\top \in \mathbb{Z}_{\geq 0}^n$ by setting $x'_i = \lambda_i$ for $i \in I$ and $x'_i = 0$ otherwise. The vector \mathbf{x}' is a solution of $A\mathbf{x} = \mathbf{0}$. Choosing $N \in \mathbb{Z}_{>0}$ large enough, we can ensure that the vector $\mathbf{x}^* + N\mathbf{x}'$ has non-negative components. Hence, $\mathbf{x} = \mathbf{x}^* + N\mathbf{x}'$ is a solution of the system $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \in \mathbb{Z}_{\geq 0}^n$ satisfying the desired sparsity estimate. The coefficients λ_i and the number N can be computed in polynomial time.

Proof (Corollary 4). The assertion follows by applying Corollary 3 for $m = 1$ and all $\tau = \{i\}$ with $i \in [n]$.

3 Proof of Theorem 6

Lemma 3. *Let $a_1, \dots, a_t \in \mathbb{Z}_{>0}$, where $t \in \mathbb{Z}_{>0}$. If $t > 1 + \log_2(a_1)$, then the system*

$$\begin{aligned} y_1 a_1 + \dots + y_t a_t &= 0, \\ y_1 \in \mathbb{Z}_{\geq 0}, y_2, \dots, y_t &\in \{-1, 0, 1\}. \end{aligned}$$

in the unknowns y_1, \dots, y_t has a solution that is not identically equal to zero.

Proof. The proof is inspired by the approach in [3, § 3.1] (used in a different context) that suggests to reformulate the underlying equation over integers as two strict inequalities and then use Minkowski's first theorem [4, Ch. VII, Sect. 3] from the geometry of numbers. Consider the convex set $Y \subseteq \mathbb{R}^t$ defined by $2t$ strict linear inequalities

$$\begin{aligned} -1 &< y_1 a_1 + \dots + y_t a_t < 1, \\ -2 &< y_i < 2 \text{ for all } i \in \{2, \dots, t\}. \end{aligned}$$

Clearly, the set Y is the interior of a hyper-parallelepiped and can also be described as $Y = \{\mathbf{y} \in \mathbb{R}^t : \|M\mathbf{y}\|_\infty < 1\}$, where M is the upper triangular matrix

$$M = \begin{pmatrix} a_1 & a_2 & \dots & a_t \\ & 1/2 & & \\ & & \ddots & \\ & & & 1/2 \end{pmatrix}.$$

It is easy to see that the t -dimensional volume $\text{vol}(Y)$ of Y is

$$\text{vol}(Y) = \text{vol}(M^{-1}[-1, 1]^t) = \frac{1}{\det(M)} 2^t = \frac{4^t}{2a_1}.$$

The assumption $t > 1 + \log_2(a_1)$ implies that the volume of Y is strictly larger than 2^t . Thus, by Minkowski's first theorem, the set Y contains a non-zero integer vector $\mathbf{y} = (y_1, \dots, y_t)^\top \in \mathbb{Z}^t$. Without loss of generality we can assume that $y_1 \geq 0$ (if the latter is not true, one can replace \mathbf{y} by $-\mathbf{y}$). The vector \mathbf{y} is a desired solution from the assertion of the lemma. \square

Proof (Theorem 6). Without loss of generality we can assume that $\gcd(\mathbf{a}) = 1$. In fact, if b is divisible by $\gcd(\mathbf{a})$ we can convert $\mathbf{a}^\top \mathbf{x} = b$ to $\bar{\mathbf{a}}^\top \mathbf{x} = \bar{b}$ with $\bar{\mathbf{a}} = \frac{\mathbf{a}}{\gcd(\mathbf{a})}$ and $\bar{b} = \frac{b}{\gcd(\mathbf{a})}$, and, if b is not divisible by $\gcd(\mathbf{a})$, the knapsack feasibility problem $\mathbf{a}^\top \mathbf{x} = b$, $\mathbf{x} \in \mathbb{Z}_{\geq 0}^n$ has no solution.

Without loss of generality, let $a_1 = \min\{a_1, \dots, a_n\}$. We need to show the existence of solution of the knapsack feasibility problem satisfying $\|\mathbf{x}\|_0 \leq 1 + \log_2(a_1)$.

Choose a solution $\mathbf{x} = (x_1, \dots, x_n)^\top$ of the knapsack feasibility problem with the property that the number of indices $i \in \{2, \dots, n\}$ for which $x_i \neq 0$ is minimized. Without loss of generality we can assume that, for some $t \in \{2, \dots, n\}$ one has $x_2 > 0, \dots, x_t > 0, x_{t+1} = \dots = x_n = 0$. Lemma 3 implies $t \leq 1 + \log_2(a_1)$. In fact, if the latter was not true, then a solution $\mathbf{y} \in \mathbb{R}^t$ of the system in Lemma 3 could be extended to a solution $\mathbf{y} \in \mathbb{R}^n$ by appending zero components. It is clear that some of the components y_2, \dots, y_t are negative, because $a_2 > 0, \dots, a_t > 0$. It then turns out that, for an appropriate choice of $k \in \mathbb{Z}_{\geq 0}$, the vector $\mathbf{x}' = (x'_1, \dots, x'_n)^\top = \mathbf{x} + k\mathbf{y}$ is a solution of the same knapsack feasibility problem satisfying $x'_1 \geq 0, \dots, x'_t \geq 0, x'_{t+1} = \dots = x'_n = 0$ and $x'_i = 0$ for at least one $i \in \{2, \dots, t\}$. Indeed, one can choose k to be the minimum among all a_i with $i \in \{2, \dots, t\}$ and $y_i = -1$.

The existence of \mathbf{x}' with at most $t - 1$ non-zero components x'_i with $i \in \{2, \dots, n\}$ contradicts the choice of \mathbf{x} and yields the assertion. \square

Acknowledgements The second author is supported by the DFG (German Research Foundation) within the project number 413995221. The third author acknowledges partial support from NSF grant 1818969.

4 Appendix

Proof (Lemma 1). Consider the prime factorization $|G| = p_1^{n_1} \dots p_s^{n_s}$. Then $|G_j| = p_1^{n_{1,j}} \dots p_s^{n_{s,j}}$ with $0 \leq n_{i,j} \leq n_i$ and, by the Chinese Remainder Theorem, the cyclic group G_j can be represented as $G_j = \bigoplus_{i=1}^s G_{i,j}$, where $G_{i,j}$ is a cyclic group of order $p_i^{n_{i,j}}$. Consequently, $G = \bigoplus_{i=1}^s \bigoplus_{j=1}^m G_{i,j}$. This is a decomposition of G into a direct sum of primary cyclic groups and, possibly, some trivial summands $G_{i,j}$ equal to $\{0\}$. We can count the non-trivial direct summands whose order is a power of p_i , for a given $i \in [s]$. There is at most one summand like this for each of the groups G_j . So, there are at most m non-trivial summands in the decomposition whose order is a power of p_i . On the other hand, the direct sum of all non-trivial summands whose order is a power of p_i is a group of order $p_i^{n_{i,1} + \dots + n_{i,s}} = p_i^{n_i}$ so that the total number of such summands is not larger than n_i , as every summand contributes the factor at least p_i to the power $p_i^{n_i}$. This shows that the total number of non-zero summands in the decomposition of G is at most $\sum_{i=1}^s \min\{m, n_i\} = \Omega_m(|G|)$. \square

Proof (Lemma 2). The proof relies on the relationship of finite Abelian groups and lattices, see [23, §4.4]. Fix a matrix $M \in \mathbb{Z}^{m \times m}$ whose columns form a basis

of A . Then $|\det(M)| = \det(A)$. There exist unimodular matrices $U \in \mathbb{Z}^{m \times m}$ and $V \in \mathbb{Z}^{m \times m}$ such that $D := UMV$ is diagonal matrix with positive integer diagonal entries. For example, one can choose D to be the Smith Normal Form of M [23, §4.4]. Let $d_1, \dots, d_m \in \mathbb{Z}_{>0}$ be the diagonal entries of D . Since U and V are unimodular, $d_1 \cdots d_m = \det(D) = \det(A)$.

We introduce the quotient group $G' := \mathbb{Z}^m / \Lambda' = (\mathbb{Z}/d_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/d_m\mathbb{Z})$ with respect to the lattice $\Lambda' := \mathcal{L}(D) = (d_1\mathbb{Z}) \times \cdots \times (d_m\mathbb{Z})$. The order of G' is $d_1 \cdots d_m = \det(D) = \det(A)$ and G' is a direct sum of at most m cyclic groups, as every $d_i > 1$ determines a non-trivial direct summand.

To conclude the proof, it suffices to show that G' is isomorphic to G . To see this, note that $\Lambda' = \mathcal{L}(D) = \mathcal{L}(UMV) = \mathcal{L}(UM) = \{Uz : z \in \Lambda\}$. Thus, the map $z \mapsto Uz$ is an automorphism of \mathbb{Z}^m and an isomorphism from Λ to Λ' . Thus, $z \mapsto Uz$ induces an isomorphism from the group $G = \mathbb{Z}^m / \Lambda$ to the group $G' = \mathbb{Z}^m / \Lambda'$. \square

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